2. a. (i) Let, (X, ρ) and (Y, d) be metric spaces and $f: X \to Y$. If $x_0 \in X$ and if $\lim_{x \to x_0} f(x)$ exists show that the limit is unique.

Proof: ~~Suppose $\lim_{x \to x_0} f(x) =$.~~

If possible, let a sequence $\{x_n\}$ in a metric space (X, d) converges to two distinct points x and y. Then $r = d(x,y) > 0$ since $x_n \to x$, $\exists\, m \in \mathbb{N}$ such that $x_n \in B(x, r/2)$, $\forall n \geq m$. Then only the finitely many terms $x_1, x_2, \dots, x_{m-1}$ can possibly belong to $B(y, r/2)$, as $B(x, r/2) \cap B(y, r/2) = \phi$

$\big(z \in B(x, r/2) \cap B(y, r/2) \Rightarrow d(x,z) < r/2$ and $d(z,y) < \frac{r}{2}$

$\Rightarrow r = d(x,y) \leq d(x,z) + d(z,y) < r$ i.e. $r < r$ — a contradiction$\big)$

Thus the sequence can not be equivalantly in $B(y, r/2)$ and Consequently, the sequence can-not converge to y, a contradiction.

Hence the limit is unique.

2. a. (ii) Let, (X, ρ) be metric space and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ where $\mathbb{R}$ is the set ~~of~~ or real no.s with usual metric, If f and g are continuous at a point $x_0 \in X$. Show that $f + g$ is also continuous at x_0.

2. c. (i) Let, (X,d) and (Y,ρ) be two metric spaces and let $f: X \to Y$ be uniformly continuous. If $\{x_n\}$ is a cauchy sequence in X, Show that $\{f(x_n)\}$ is cauchy in Y. Does result always hold if f is continuous? Support your ans.

Ans: Let, $\varepsilon > 0$ be preassigned. Then by uniform continuity of f, $\exists\ \delta > 0$ such that $x, x' \in X$ and $d(x, x') < \delta \Rightarrow$

$\Rightarrow \rho(f(x), f(x')) < \varepsilon$.

Now, $\{x_n\}$ is Cauchy in $X \Rightarrow \exists\ K \in \mathbb{N}$ such that $d(x_n, x_m) < \delta$, $\forall\ n, m \geq K$

$\Rightarrow \rho(f(x_n), f(x_m)) < \varepsilon$, $\forall\ m, n \geq K$.

Hence $\{f(x_n)\}$ is a cauchy Sequence in Y.

5: a. (i) Define a compact set and a sequentially compact set in a metric space. Show that a set S is compact iff S is sequentially compact.

(ii) Show that every compact subset of a metric space is closed.

Ans: (i) ■ <u>Compact</u>: A metric space (X, d) is said to be a compact metric space if and only if every open cover of X admits a finite subcover. i.e. $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ is a family of open sets in (X, d) satisfing $\cup\{A_\alpha : \alpha \in \Delta\} = X$, then $\exists$ a finite family $\{A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}\}$ from $\mathcal{A}$ such that $\cup\{A_{\alpha_i} : i = 1, 2, \dots, n\} = X$.

■ <u>Sequentially Compact</u>: A metric space (X, d) is called sequentially compact if every sequence in X produce a convergent subsequence.

e.g. The space $\mathbb{R}$ of reals is not sequentially compact because a sequence of natural numbers has no convergent subsequence.

■ Let, (X, d) be compact and $\{x_n\}$ be any sequence in (X, d). Consider $G_n = \{x_n, x_{n+1}, \dots\}$ for $n = 1, 2, 3, \dots$

Clearly, $G_n \supset G_{n+1}$, for $n = 1, 2, \dots$. So, $\overline{G_n} \supset \overline{G_{n+1}}$

~~Then~~ Thus $\{\overline{G_n}\}$ is a decreasing chain of non-empty closed sets in (X, d).

Thus $\{\overline{G_n}\}$, $n=1,2,3,\cdots$ is a family of closed set with FIP.
By compactness of (S,d), we have $\bigcap_{n=1}^{\infty}\overline{G_n} \neq \phi$.
Let us take $u \in \bigcap_{n=1}^{\infty}\overline{G_n}$.
Let, $\epsilon>0$ be given and choose k_0 such that $\frac{1}{k}<\epsilon$, for $k \geq k_0$.
Then $\forall n$ we have $u \in \bigcap_{n=1}^{\infty}\overline{G_n}$ and hence $\exists\, x_{n_k} \in G_n$ satisfying
$d(u, x_{n_k}) < \frac{1}{k}$
$< \epsilon$, for $k \geq k_0$
i.e. $\lim_k x_{n_k} = u$.
Hence $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$.
Hence, the theorem.

Conversely let, (S,d) be a sequentially compact metric space.
From sequentially compact metric space is Lindelöf, every open cover of ~~A~~ S has a countable subcover.
Let, C be an open cover ~~A~~ S.
Extract from it is a countable subcover $\{U_1, U_2, \cdots\}$.
Aiming for a contradiction, suppose that $\exists$ no finite subcover of C.
Then for all $n (\geq 1) \in \mathbb{N}$, the set $\{U_1, U_2, \cdots, U_n\}$ does not cover S.
Hence, it is possible to choose $x_n \in S$ such that $x_n \notin U_1 \cup U_2 \cup \cdots \cup U_n$
Thus we constract a infinite sequence $\{x_n\}_{n\geq 1}$ of points of ~~A~~ S.
By assumption S is sequentially compact.
Thus, $\{x_n\}_{n\geq 1}$ has a subsequence which converges to some $x \in S$.
But because the $U_i : i \geq 1$ forms a ~~...~~ cover for S, $\exists$ some U_m such that $x \in U_m$.
Thus there is an infinite no. of terms in the sequence $\langle x_i \rangle$ which are contained in U_m. But from the method of constraction of $\langle x_i \rangle$ each U_n can ~~...~~ contain only points x_i with $i \leq n$.
i.e. each U_n can only contain a finite no. of terms of $\langle x_i \rangle$
This is a contradiction. Thus our assumption that ~~there~~ $\nexists$ any finite sub cover of C was false.
Hence, the result.

(ii) Suppose X is a metric space. Let $A \subset X$ be a compact subset of X and let $\{V_\alpha\}$ be an open cover of A. Then there are finitely many indices α_i such that $A \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n}$
Now let x be a limit point of A. Assume $x \notin A$. If $x \notin A$

put, $\delta = \inf\{d(x,y) : y \in A\}$.
Take $\epsilon = \frac{\delta}{2}$, then $B_d(x,\epsilon) \cap A = \phi$ so that a neighbourhood of x does not intersect A asserting that x can-not be a limit point of A, hence $x \in A$. So that A is closed.

5. b. (i) Show that in a metric space every complete subset is closed. (5)

Ans: ~~Let~~, X be a metric space, and let Y be a complete subspace of X. Then we have to show that Y is closed.

Let, $x \in \bar{Y}$ be a point in the closure of Y. Then by defination of closure, from each ball $B(x, 1/n)$ centered in x, we can select a point $y_n \in Y$. This is clearly a Cauchy sequence in Y, and its limit is x. Hence by the completeness of Y, $x \in \bar{Y}$ and Thus $Y = \bar{Y}$. Thus the result.

b. (ii) Prove that the intersection of any collection of complete subsets of a metric space is complete and the union of finite ~~no. of~~ collection of complete subsets of a metric space is complete.

Proof: ■ ~~Let, $\{A_i\}_{i\in I}$ be any collection of co~~
Let, $\{x_n\}$ be a Cauchy sequence in $\bigcap_{i\in I} A_i$, an intersection of any collection $\{A_i\}_{i\in I}$ of complete subsets of X. Since $\{x_n\}$ is a Cauchy sequence in $\bigcap_{i\in I} A_i$, it is Cauchy sequence in A_1, A_2.
Since, A_1 is complete, $x_n \to x$, where $x \in A_1$
Similarly $\{x_n\}$ is cauchy sequence in A_2, Since A_2 Is complete then $x \in A_2$. Without loss of generality, $x \in A_i$, $\forall\, i \in I$.
$\therefore x \in \bigcap_{i\in I} A_i$. Hence $\bigcap_{i\in I} A_i$, then intersection of any collection of complete subsets of X is complete.

■ Let, $S = S_1 \cup S_2 \cup \dots \cup S_n$. be the union of a finite no. of complete subsets of X. Choose a Cauchy Sequence $\{x_n\} \in S$, then there is some $K \in \{1, 2, \dots, n\}$ such that infinitely many x_n's $\in S_K$. (Otherwise the entire sequence would be a finite set, which is the trivial case we can ignore).
Thus there is a subsequence $\{x_{n_j}\} \in S_K$ which must have a limit in S_K (Since S_K is complete). However if a subsequence of

a Cauchy sequence converges to a point, then the entire sequence converges to that point.
Hence that point in S. Thus S is complete.
Hence, the theorem.

5b (iii) Give an example to show that the union of finite collection of complete subsets may not be complete. (2)

2.b. (ii) Let (X, d) be a metric space and let $x_1, x_2 \in X$. Prove that there is a neighbourhood N_1 of x_1 and a neighbourhood N_2 of x_2 such that $N_1 \cap N_2 = \emptyset$.

Proof: we have $x_1 \in X$ as well as $x_2 \in X$ are two neighbourhoods of N_1 and N_2, then $\exists$ two (+ve) reals r_1 and r_2 such that

$$x_1 \in S(x_1, r_1) \subset N_1 \quad \text{and} \quad x_2 \in S(x_2, r_2) \subset N_2.$$

(ii) Define an open set in a metric space. Prove that the intersection of a finite no. of open sets in a metric space is open. Show by an example that this result is not true for infinite no. of open sets.

(2+1=3)

Ans: Open sets: A subset G of a metric space (X,d) is called an open set if every point of G is an interior point of G.

■ Let Δ be ~~the~~ a finite index set such that $\forall \alpha \in \Delta$, the set A_α is open in the metric space (X,d). We have to prove that the set $A = \bigcap_{\alpha \in \Delta} A_\alpha$ is also open in (X,d).

If $\Delta = \phi$, then $A = X$, which is open.

If $\Delta \neq \phi$, but the sets $A_\alpha, \alpha \in \Delta$ are mutually disjoint, then $A = \bigcap_{\alpha \in \Delta} A_\alpha = \phi$, which is open in (X,d).

Finally let $A \neq \phi$ and $\Delta = \{1, 2, \dots, n-1, n\}$. Then

$$x \in A \Rightarrow x \in A_\alpha, \forall \alpha = 1, 2, \dots, n.$$

So, $\exists$ +ve reals $r_1, r_2, \dots, r_n$ such that $x \in S(x, r_\alpha) \subset A_\alpha$ for all $\alpha = 1, 2, \dots, n$. Let, $r = \min\{r_1, r_2, \dots, r_n\}$. Clearly $r > 0$ and $S(x,r) \subset S(x, r_\alpha), \forall \alpha = 1, 2, \dots, n$.

And hence we can write $x \in S(x,r) \subset S(x, r_\alpha) \subset A_\alpha, \forall \alpha \in \Delta$

$$\Rightarrow x \in S(x,r) \subset \bigcap_{\alpha \in \Delta} A_\alpha = A.$$

Thus every point of A are its interior point in (X,d).

$\therefore$ The set A is open in the metric space (X,d).

■ Let, $A_n = (-n, n), \forall n \in \mathbb{N}$, Then $A = \bigcap_{n \in \mathbb{N}} A_n = (-1, 1)$, which is open. But

$C_n = \left(-1-\frac{1}{n}, 1+\frac{1}{n}\right), \forall n \in \mathbb{N}$. Then $C = \bigcap_{n \in \mathbb{N}} C_n = [-1, 1]$. But C is not open. Since the points -1 and 1 are not interior points of the set C in this metric space.